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The Fuzzy Ant

Abstract: We apply fuzzy modeling to transform a verbal description of the foraging behavior of ants into a well-defined mathematical model. The resulting model is simpler, more plausible, and more amenable to analysis than previously suggested models.

We believe that fuzzy modeling may be suitable for addressing biomimicry, that is, the development of artificial products or machines that mimic biological phenomena, in a systematic manner.

Keywords: Linguistic modeling, social insects, mass foraging, Hopfield-type neural networks, soft computing, emergent behavior, differential equations with time delay, Lyapunov-Krasovskii functional.

1. Introduction

Mathematical models are indispensable when we wish to rigorously analyze dynamic systems. Such a model summarizes and interprets the empirical data. It can also be used to simulate the system on a computer and to provide predictions for future behavior. Mathematical models of the atmosphere, which can be used to provide weather predictions, are a classic example.

In physics, and especially in classical mechanics, it is sometimes possible to derive mathematical models using *first principles* such as Euler-Lagrange equations [1]. In other fields of science, like biology, economics and psy-

chology, no such *first principles* are known. In many cases, however, researchers have provided descriptions and explanations of various phenomena stated in *natural language*. Science can greatly benefit from transforming these verbal descriptions into mathematical models. This raises the following problem.

Problem 1 Find an efficient way for transforming verbal descriptions into a mathematical model or computer algorithm.

This problem was already addressed in the field of *artificial expert systems* (AESs). These are computer algorithms that emulate the functioning of a human expert, for example, a physician who can diagnose diseases or an operator who can successfully control a specific system. One approach for constructing AESs is based on questioning the human expert in order to extract information on his/her functioning. This leads to a verbal description, which must be transformed into a computer algorithm.

Fuzzy modeling (FM) is routinely used by knowledge engineers to construct AESs. The knowledge extracted from the human expert is stated as a collection of If-Then rules expressed using natural language. Defining the verbal terms in the rules using suitable membership functions, and inferring the rule base, yields a well-defined mathematical model. Thus, the verbal information is transformed into a form that can be programmed on a computer. This approach was used to develop AESs that diagnose diseases, control various processes, and much more [2]–[4].

The overwhelming success of fuzzy expert systems suggests that FM may

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be a suitable approach for solving Problem 1. Indeed, the real power of fuzzy logic theory is in its ability to efficiently handle and manipulate verbally-stated information (see, e.g., [5]–[8]).

Recently, FM was applied in a different context, namely, in transforming verbal descriptions and explanations of *natural phenomena* into a mathematical model. Indeed, FM provides a simple and efficient means for transforming the researcher's understanding, stated in words, into a rigorous mathematical model.

The applicability and usefulness of this approach was demonstrated using examples from the field of ethology: (1) territorial behavior in the stickleback [9], as described by Nobel Laureate Konrad Lorenz in [10]; (2) the mechanisms governing the orientation to light in the planarian *Dendrocoleum lacteum* [11]; and (3) the self-regulation of population size in blow-flies [12].

There are several reasons that our work focuses on models from ethology. First, many animal (and human) actions are “fuzzy.” For example, the response to a (low intensity) stimulus might be what Heinroth called *intention movements*, that is, a slight indication of what the animal is tending to do. Tinbergen [13, Ch. IV] states: “As a rule, no sharp distinction is possible between intention movements and more complete responses; they form a continuum.”¹ Hence, FM seems an appropriate tool for studying such behaviors.

The second reason is that studies of animal behavior often provide a *verbal* description of both field observations and interpretations. For example, Fraenkel and Gunn describe the behavior of a cockroach that becomes stationary when a large part of its body surface is in contact with a solid object as: “A high degree of contact causes low activity ...” [15, p. 23]. Note that this can be immediately stated as the fuzzy rule: If degree of contact is *high*, then activity is *low*. In fact, it is customary to describe the behavior of simple organisms using simple rules of thumb [16].

Another reason is that considerable research is currently devoted to the field of *biomimicry*—the development of artificial products or machines that mimic biological phenomena [17]–[20]. Over the course of evolution, living systems developed efficient and robust solutions to various problems. Some of these problems are also encountered in engineering applications.

For example, plants had to develop efficient mechanisms for absorbing and utilizing solar energy. Engineers that design solar cells face a very similar challenge.

Thus, the design of artificial systems may be inspired by the behavior of living systems.

An important component in biomimicry is the ability to perform reverse engineering of the functioning of a living system. We believe that FM may be suitable for addressing biomimicry in a systematic manner. Namely, start with a verbal description of the biological system’s behavior (e.g., foraging in ants) and, using fuzzy logic theory, obtain a mathematical model of this behavior that can be immediately implemented by artificial systems (e.g., autonomous robots).

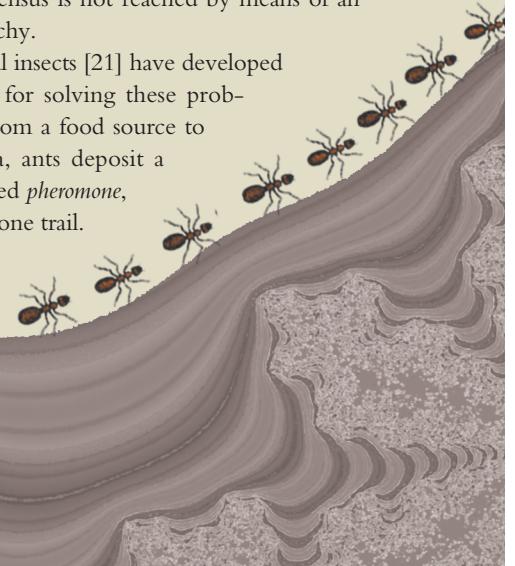
In this paper, we take a first step in this direction by using FM to develop a mathematical model for the foraging behavior of ants. The resulting model is simpler, more plausible, and more amenable to analysis than previously suggested models. Simulations and rigorous analysis of the resulting model show that it is congruent with the behavior actually observed in nature. Furthermore, the new model establishes an interesting link between the averaged behavior of a colony of foraging ants and mathematical models used in the theory of artificial neural networks (see Section 7 below).

The next section reviews the foraging behavior of ants as described by several researchers. Section 3 applies FM to transform the verbal description into a simple mathematical model describing the behavior of a single ant. In Section 4, this is used to develop a stochastic model for the behavior of a colony of identical ants. Section 5 reviews an averaged model of the colony. Sections 6 and 7 are devoted to studying this averaged model using simulations and rigorous analysis, respectively. The final section concludes. All the proofs are collected in the Appendix.

2. Foraging Behavior of Ants

A foraging animal may have a variety of potential paths to a food item. Finding the shortest path minimizes time, effort, and exposure to hazards. For mass foragers, such as ants, it is also important that *all* foragers reach a consensus when faced with a choice of paths. This is not a trivial task, as ants have very limited capabilities of processing and sharing information. Furthermore, this consensus is not reached by means of an ordered chain of hierarchy.

Ants and other social insects [21] have developed an efficient technique for solving these problems. While walking from a food source to the nest, or vice versa, ants deposit a chemical substance called *pheromone*, thus forming a pheromone trail.



¹It is interesting to recall that Zadeh [14] defined a fuzzy set as “a class of objects with a continuum of grades of membership.”

The real power of fuzzy logic theory is in its ability to efficiently handle and manipulate verbally-stated information.

Following ants are able to smell this trail. When faced by several alternative paths, they tend to choose those that have been marked by pheromones. This leads to a positive feedback mechanism: a marked trail will be chosen by more ants that, in turn, deposit more pheromone, thus stimulating even more ants to choose the same trail.

Goss et al. [22] designed an experiment in order to study the behavior of the Argentine ant *Iridomyrmex humilis* while constructing a trail around an obstacle. A laboratory nest was connected to a food source by a double bridge (see Figure 1). Ants leaving the nest or returning from the food item to the nest must choose a branch. After making the choice, they mark the chosen branch. Ants that take the shorter of the two branches return sooner than those using the long branch. Thus, in a given time unit, the short branch receives more markings than the long branch. This small difference in the pheromone concentrations is amplified by the positive feedback process. The process generally continues until nearly all the foragers take the same branch, neglecting the other one. In this sense, it appears that the entire colony has decided to use the short branch.

The positive feedback process is counteracted by negative feedback due to pheromone evaporation. This plays an important role: the markings of obsolete paths, that lead to depleted food sources, disappear. This increases the chances of detecting new and more relevant paths.

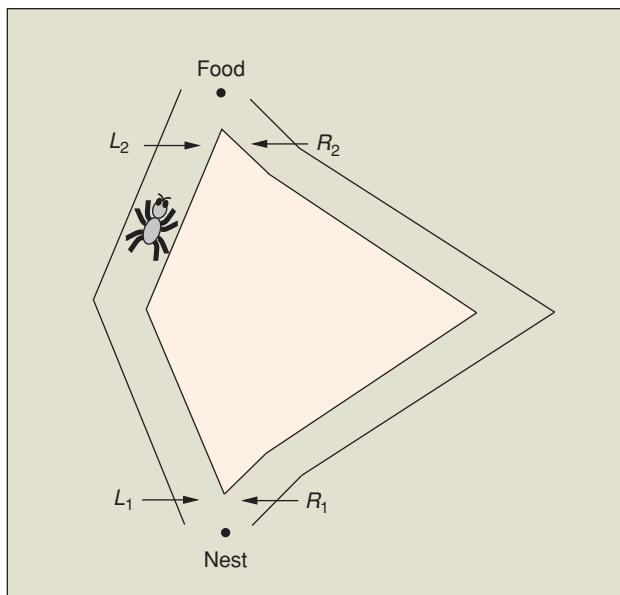


FIGURE 1 Experimental setup with two branches: the left branch is shorter.

Note that in this model, no single ant compares the length of the two branches directly. Furthermore, the ants do not communicate directly. Rather, they change the environment by laying pheromone trails and thus indirectly affecting the behavior of other ants. The net result, however, is that the entire colony appears to have made a well informed choice of using the shorter branch.

The fact that simple individual behaviors can lead to a complex *emergent behavior* has been known for centuries. King Solomon marveled at the fact that “the locusts have no king, yet go they forth all of them by bands” (Proverbs 30:27). More recently, it was noted that this type of emergent collective behavior is a desirable property in many artificial systems. From an engineering point of view, the solution of a complex problem using simple agents is an appealing idea, which can save considerable time and effort. Furthermore, the specific problem of detecting the shortest path is important in many applications, including robot navigation, graph theory and communication engineering (see e.g., [23]–[25]).

3. Fuzzy Modeling

In this section, we apply FM [26] to transform the verbal descriptions into a mathematical model. The approach consists of the following stages: (1) identification of the variables; (2) stating the verbal information as a set of fuzzy rules relating the variables; (3) defining the fuzzy terms using suitable membership functions; and (4) inferring the rule base to obtain a mathematical model [9].

When creating a mathematical model from a verbal description there are always numerous degrees of freedom. In the FM approach, this is manifested in the freedom in choosing the components of the fuzzy model: the type of membership-functions, logical operators, inference method, and the values of the different parameters. The following guidelines may be helpful in selecting the different components of the fuzzy model (see also [27] for details on how the various elements in the fuzzy model influence its behavior).

First, it is important that the resulting mathematical model has the simplest possible form, in order to be amenable to analysis. Thus, for example, a Takagi-Sugeno model with singleton consequents might be more suitable than a model based on Zadeh’s compositional rule of inference.

Second, when modeling real-world systems, the variables are physical quantities with dimensions (e.g., length, time). *Dimensional analysis* [28], [29], the process of introducing dimensionless variables, can often simplify the resulting equations and decrease the number of parameters.

Third, sometimes the verbal description of the system is accompanied by measurements of various quantities in the system. In this case, methods such as fuzzy clustering, neural learning, or least squares approximation (see, e.g., [30]–[32] and the references therein) can be used to fine-tune the model using the discrepancy between the measurements and the model’s output.

For the foraging behavior in the simple experiment described above, we need to model the choice-making process of an ant facing a fork in a path. We use the following verbal description [33]: “If a mass forager arrives at a fork in a chemical recruitment trail, the probability that it takes the left branch is all the greater as there is more trail pheromone on it than on the right one.”

An ant is a relatively simple creature, and any biologically feasible description of its behavior must also be simple, as is the description above. Naturally, transforming this description into a set of fuzzy rules will lead to a simple rule-base. Nevertheless, we will see below that the resulting fuzzy model, although simple, has several unique advantages.

3.1 Identification of the Variables

The variables in the model are the pheromone concentrations on the left and right branches denoted L and R , respectively. The output is $P = P(L, R)$, which is the probability of choosing the left branch.

3.2 The Fuzzy Rules

According to the verbal description given above, the probability P of choosing the left branch at the fork is directly correlated with the difference in pheromone concentrations $D := L - R$.

We state this using two fuzzy rules:

- If D is positive Then $P = 1$.
- If D is negative Then $P = 0$.

3.3 The Fuzzy Terms

A suitable membership function for the term *positive*, $\mu_{pos}(\cdot)$, must satisfy the following constraints: $\mu_{pos}(D)$ is a monotonically increasing function, $\lim_{D \rightarrow -\infty} \mu_{pos}(D) = 0$, and $\lim_{D \rightarrow \infty} \mu_{pos}(D) = 1$. There are good reasons for using the hyperbolic tangent function in both artificial neural networks and fuzzy models [34], so we use the membership function $\mu_{pos}(D) := (1 + \tanh(qD))/2$. The parameter $q > 0$ determines the slope of $\mu_{pos}(D)$. The term *negative* is modeled using $\mu_{neg}(D) := 1 - \mu_{pos}(D)$. As we will see below, this choice also leads to a mathematical model for the behavior of a colony of ants that is better than previously suggested models.

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3.4 Fuzzy Inferencing

We use center of gravity inference (see, e.g., [27]). This yields

$$\begin{aligned} P(D) &= \mu_{pos}(D)/(\mu_{pos}(D) + \mu_{neg}(D)) \\ &= (1 + \tanh(qD))/2. \end{aligned} \quad (1)$$

Note that $P(D) \in (0, 1)$ for all $D \in \mathbb{R}$.

3.5 Parameter Estimation

Goss et al. [22] and Deneubourg et al. [35] suggested the following function:

$$P_{n,k}(L, R) = (k + L)^n / ((k + L)^n + (k + R)^n). \quad (2)$$

As noted in [35], the parameter n determines the degree of nonlinearity of $P_{n,k}$. The parameter k corresponds to the degree of attraction attributed to an unmarked branch: as k

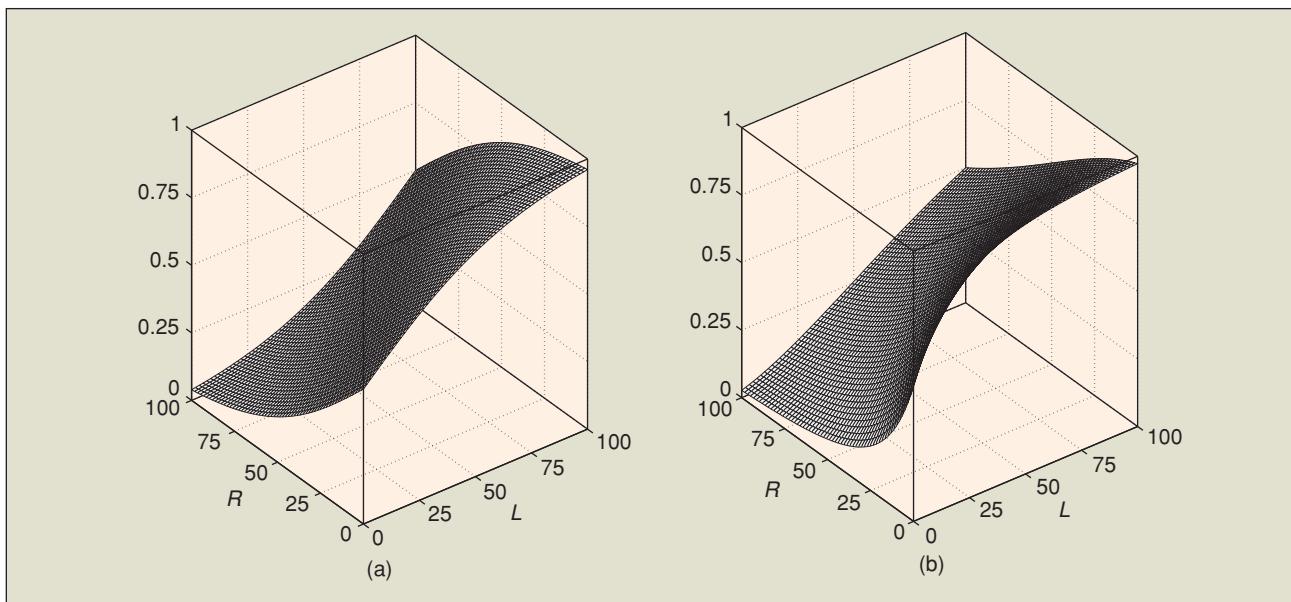


FIGURE 2 The functions: (a) $P(L - R)$ with $q = 0.016$; (b) $P_{2,20}(L, R)$.

Fuzzy modeling approach may be suitable for addressing biomimicry in a systematic manner. Namely, start with a verbal description of the biological system's behavior (e.g., foraging in ants) and, using fuzzy logic theory, obtain a mathematical model of this behavior that can be immediately implemented by artificial systems (e.g., autonomous robots).

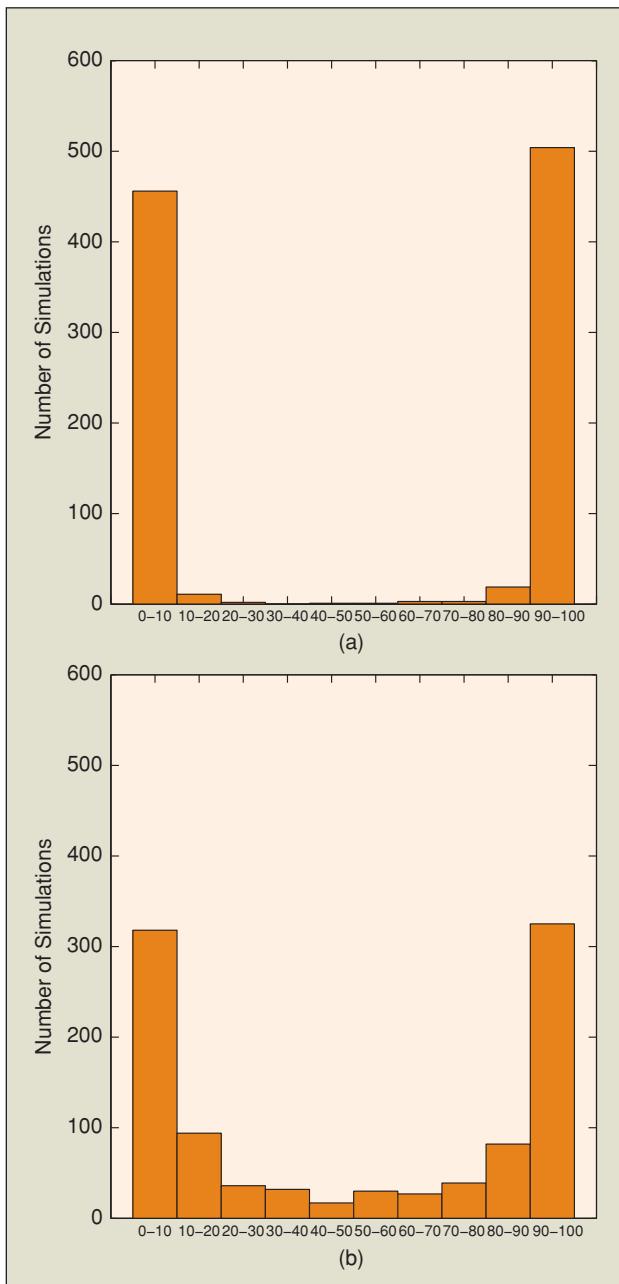


FIGURE 3 Percentage of ants that chose the left branch, for $r = 1$ and $s = 0.01$. (a) Using $P(L - R)$ with $q = 0.016$. (b) Using $P_{2,20}(L, R)$.

increases, a greater marking is necessary for the choice to become significantly nonrandom.

Note that for $L \gg R$ ($L \ll R$) both (1) and (2) yield that the probability of choosing the left branch goes to one (zero). Our model is simpler and seems more plausible, as it depends only on the difference $D = L - R$, and it includes only a single parameter.

Deneubourg et al. [35] found that for $n = 2$ and $k = 20$ the function (2) agrees well with the actual behavior observed in an experiment involving *Iridomyrmex humilis*. Deneubourg et al. do not provide the exact biological data they used. In order to obtain (indirectly) a reasonable match with the real biological behavior, we tried to match $P(D)$ with the function $P_{2,20}(L, R)$. To do so, we (numerically) solved the problem: $\min_q \sum_{(L,R) \in A} |P(L - R) - P_{2,20}(L, R)|^2$, where $A = [0, 1, \dots, 100] \times [0, 1, \dots, 100]$. The best match is obtained for $q = 0.016$ (see Figure 2).

In the next sections, we simulate and rigorously analyze the behavior of a colony of “fuzzy” ants, that is, ants that choose between two alternative paths according to the function $P(D)$.

4. The Stochastic Model

We model the scenario depicted in Figure 1 as a sequence of stochastic events. Initially, at time $t = 0$, all trails are unmarked: $L_1(0) = R_1(0) = L_2(0) = R_2(0) = 0$. Let τ denote the time needed to travel from the nest to the food item using the left branch. The corresponding time for the right branch is $r\tau$, with $r \geq 1$.

At every time step $t = 0, 1, \dots, 1000$, a new ant heads out of the nest and chooses a branch at the fork near the nest. The choice is made according to the probability $P(L_1(t), R_1(t))$. If the choice is to follow the left [right] branch then $L_1[R_1]$ is increased by 1. This ant reaches the fork near the food source at time $t + \tau[t + r\tau]$, adding 1 to $L_2(t + \tau)[R_2(t + r\tau)]$, and then chooses which branch to use on its return according to $P(L_2(t + \tau), R_2(t + \tau))[P(L_2(t + r\tau), R_2(t + r\tau))]$. Consequently, either L_2 or R_2 is increased, and after τ or $r\tau$ time steps, 1 is added to L_1 or R_1 , respectively.

The effect of pheromone evaporation, with rate $s \in (0, 1]$, is modeled by setting $L_i(t+1) = (1-s)L_i(t)$, $R_i(t+1) = (1-s)R_i(t)$ at every time step.

To estimate the traffic at steady-state, we numbered the left/right decisions consecutively, and the results presented below are based on decisions 501–1000.

Figure 3 summarizes the results of 1000 simulations with $\tau = 20$, $s = 0.01$, and $r = 1$ (equal branches). Using (1), almost all simulations end up with the colony choosing one of the two branches. In 523 simulations 80–100% of the ants end up choosing the left branch. In almost all other simulations 80–100% of the ants end up choosing the right branch.

These results seem to agree with the behavior actually observed in experiments using double bridge with equal length branches: “Should more ants use one of the branches at the beginning of the experiment, either by chance or for some other reason, then that branch will be most strongly marked and attract more ants, and so on until most of the ants use that branch” [33, p. 403].

Similar behavior is seen when using the probability function (2), but the distribution is more “blurry”, as there are considerably more simulations ending with no clear-cut choice of one of the branches.

Figure 4 summarizes the results of 1000 simulations with $\tau = 20$, $s = 0.01$, and $r = 2$, that is, the time needed to follow the right branch is twice as long as that of the left branch. It may be seen that using the probability function (1) leads to a clear-cut distribution: in 807 simulations 80–100% of the ants choose the shorter branch. In 186 simulations, 0–20% of the ants choose the shorter branch. Thus, in 993 of the 1000 simulations the colony converges to a favorable branch, and in 81% of the simulations this is indeed the shorter branch.

These results agree with the behavior actually observed in nature: “The experiments show that *L. niger* colonies nearly always select the shorter of two branches, and do so with a large majority of foragers” [33, p. 413].

Using the probability function (2) leads again to a more “blurry” distribution, where in more simulations there is no clear convergence towards a favorable branch.

5. The Averaged Model

Following [22], we now consider a deterministic model that describes the “average” concentration of pheromones in the system. This averaged model is a set of four nonlinear delay differential equations (DDEs):

$$\begin{aligned}\dot{L}_1(t) &= FP_1(t) + FP_2(t - \tau) - sL_1, \\ \dot{L}_2(t) &= FP_2(t) + FP_1(t - \tau) - sL_2, \\ \dot{R}_1(t) &= F(1 - P_1(t)) + F(1 - P_2(t - r\tau)) - sR_1, \\ \dot{R}_2(t) &= F(1 - P_2(t)) + F(1 - P_1(t - r\tau)) - sR_2,\end{aligned}\quad (3)$$

where $P_1[P_2]$ is the probability of choosing the left branch at fork 1 [fork 2] as defined in (1), and F is the number of ants per second leaving the nest. The first equation can be explained as follows: the change in the pheromone concentration $L_1(t)$ is due to: (1) the ants that choose to use the left branch at fork 1 and deposit pheromone as they start going; (2) the ants that choose the left branch at point 2 at time $t - \tau$. These reach point 1, and deposit pheromone at the left branch, after τ sec; and (3) the reduction of pheromone due to evaporation. The other equations follow similarly.

Eq. (3) is similar to the model used in [22], [33], but we use P rather than $P_{n,k}$. It turns out that the fact that $P = P(L - R)$ allows us to transform the averaged model

The net result, however, is that the entire colony appears to have made a well informed choice of using the shorter branch.

into a two-dimensional model, which is easier to analyze. Furthermore, using P leads to a novel and interesting link between the averaged behavior of the ant colony and mathematical models used in the theory of artificial neural networks (see Section 7 below).

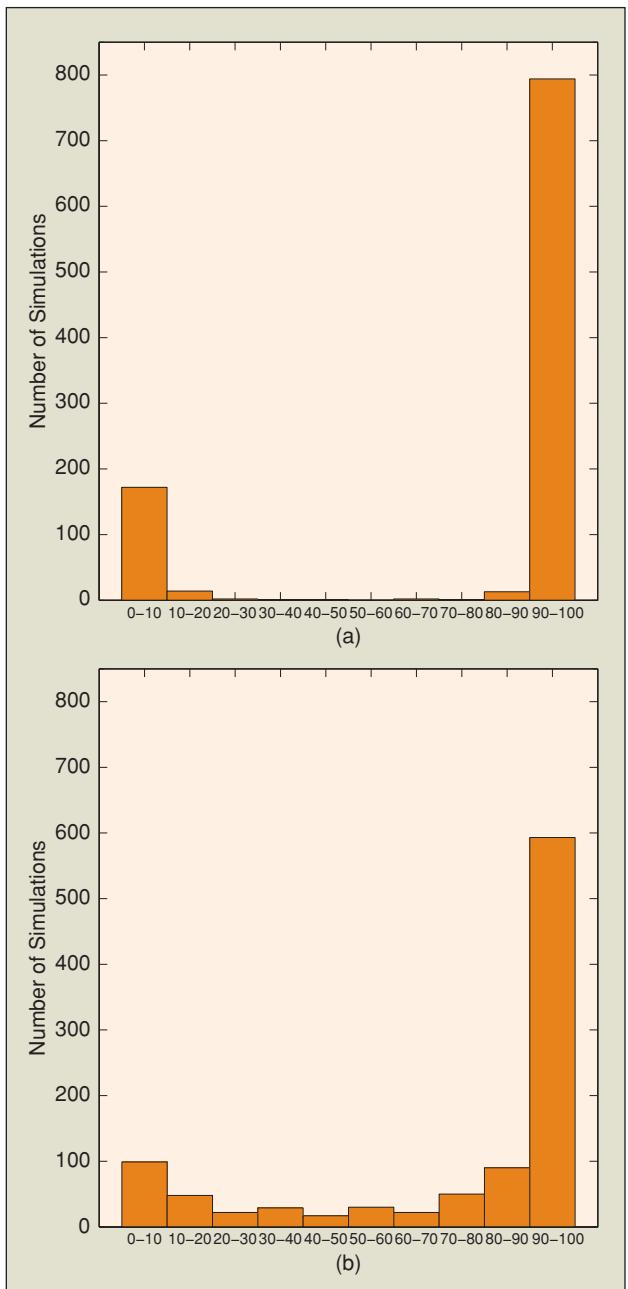


FIGURE 4 Percentage of ants that chose the left branch, for $r = 2$ and $s = 0.01$. (a) Using $P(L - R)$ with $q = 0.016$. (b) Using $P_{2,20}(L, R)$.

If a mass forager arrives at a fork in a chemical recruitment trail, the probability that it takes the left branch is all the greater as there is more trail pheromone on it than on the right one.

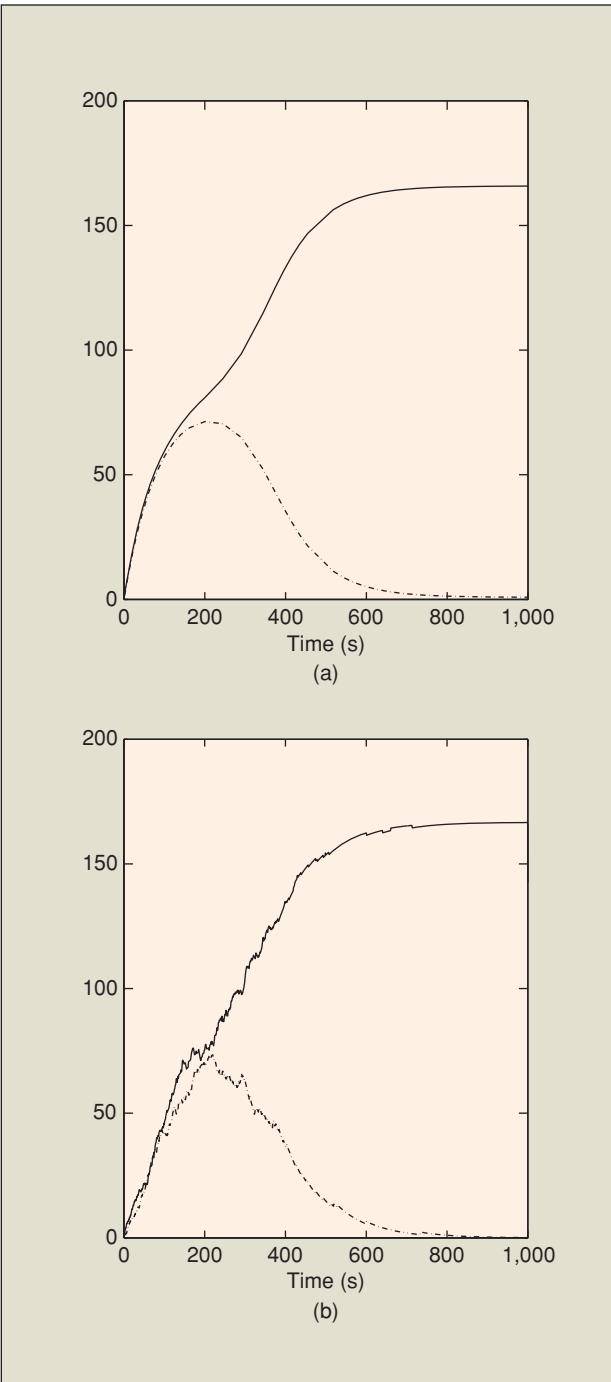


FIGURE 5 $L_1(t)$ (solid line) and $R_1(t)$ (dashed line) for $r = 2, s = 0.012$. (a) Solution of DDE (3) with $q = 0.016$, (b) Monte Carlo simulation.

6. Simulations

We simulated (3) and compared the results to a Monte Carlo simulation of the stochastic model. Note that (3) describes pheromone concentrations, not ant numbers. Yet there is, of course, a correspondence between these two since the pheromones are laid by ants.

We used the parameters $F = 1, \tau = 20, r = 2$, and the initial conditions $L_1(t) = 1, L_2(t) = R_1(t) = R_2(t) = 0$, for all $t \in [-r\tau, 0]$. To analyze the effect of evaporation, we considered two values of s . For each value, we simulated the stochastic and averaged models with a colony of 1000 foragers.

Figure 5 depicts the pheromone concentrations as a function of time for $s = 0.012$. The stochastic model and the averaged model behave similarly. Initially, the concentrations on both branches are equal. As time progresses, the left branch, which is the shorter one, receives more and more markings. The difference $L_1(t) - R_1(t)$ converges to a steady-state value of 164.98.

Figure 6 depicts the results of the simulations for a higher value of evaporation rate, namely, $s = 0.1$. In this case, the traffic tends to distribute equally along the two branches. The reason is that the positive feedback process of pheromone laying is ineffective because the pheromones evaporate faster than the ants can lay them. This makes it impossible to detect the shorter branch. This is in agreement with the behavior actually observed in nature: “Only if the amount of ants arriving at the fork is insufficient to maintain the pheromone trail in the face of evaporation will no choice be arrived at and the traffic distributed equally over the two branches” [33, p. 408].

7. Analysis of the averaged model

Let $v_j(t) := (L_j(t) - R_j(t))/F, j = 1, 2$, that is, the scaled difference between the pheromone concentrations on the left-hand and right-hand sides of fork j . Using (3) and (1) yields

$$\begin{aligned} \dot{v}_1(t) &= -sv_1(t) + \tanh(pv_1(t)) + (\tanh(pv_2(t-\tau)) \\ &\quad + \tanh(pv_2(t-r\tau)))/2, \\ \dot{v}_2(t) &= -sv_2(t) + \tanh(pv_2(t)) + (\tanh(pv_1(t-\tau)) \\ &\quad + \tanh(pv_1(t-r\tau)))/2, \end{aligned} \quad (4)$$

where $p := qF > 0$. Note that this simplification from a fourth-order to a second-order model is possible because our probability function, unlike (2), depends only on the difference $L - R$.

Models in the form (4) were used in the context of Hopfield-type artificial neural networks (ANNs) with time-delays (see [36] and the references therein). In this context, (4) represents a system of two dynamic neurons, each possessing nonlinear feedback, and coupled via nonlinear time delayed connections. This yields an interesting and novel connection between the aggregated behavior of the colony and classical models used in the theory of ANNs. The set of ants choosing the left (right) path corresponds to the state of the first (second) neuron. The effect of the chemical

communication between the ants corresponds to the time-delayed feedback connections between the neurons.

7.1 Equilibrium Solutions

The equilibrium solutions of (4) are $\mathbf{v}(t) \equiv (\nu, \nu)^T$, where ν satisfies

Should more ants use one of the branches at the beginning of the experiment, either by chance or for some other reason, then that branch will be most strongly marked and attract more ants, and so on until most of the ants use that branch.

$$s\nu - 2 \tanh(p\nu) = 0. \quad (5)$$

The properties of the hyperbolic tangent function yield the following result.

Proposition 1 If $s > 2p$ then the unique solution of (5) is $\nu = 0$, so (4) admits a unique equilibrium solution $\mathbf{v}(t) \equiv \mathbf{0}$. If $s \in (0, 2p)$ then (3) admits three solutions $0, \underline{\nu}, -\underline{\nu}$, with $\underline{\nu} > 0$, and (4) admits three equilibrium solutions: $\mathbf{v}(t) \equiv \mathbf{0}$, $\mathbf{v}(t) \equiv \mathbf{v}^1 := (\underline{\nu}, \underline{\nu})^T$, and $\mathbf{v}(t) \equiv -\mathbf{v}^1$.

7.2 Stability

For the sake of completeness, we recall the necessary stability definitions (for more details, see [37]–[39]). Consider the DDE

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{x}(t-d)), \quad t \geq t_0, \quad (6)$$

with the initial condition $\mathbf{x}(t) = \phi(t)$, $t \in [t_0 - d, t_0]$, and suppose that $\mathbf{x}(t) \equiv \mathbf{0}$ is an equilibrium solution. For a continuous function $\phi : [t_0 - d, t_0] \rightarrow \mathbb{R}^n$, define the continuous norm by $\|\phi\|_c := \max\{\|\phi(\theta)\| : \theta \in [t_0 - d, t_0]\}$.

Definition 1 The solution $\mathbf{0}$ is said to be uniformly stable if for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|\phi\|_c < \delta$ implies that $\|\mathbf{x}(t)\| < \epsilon$ for $t \geq t_0$. It is uniformly asymptotically stable if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\alpha > 0$, there exists $T = T(\delta_a, \alpha)$, such that $\|\phi\|_c < \delta_a$ implies that $\|\mathbf{x}(t)\| < \alpha$ for $t \geq t_0 + T$. It is globally uniformly asymptotically stable (GUAS) if it is uniformly asymptotically stable and δ_a can be an arbitrarily large number. \square

Proposition 1 suggests that we need to consider the two cases $s > 2p$ and $s \in (0, 2p)$ separately.

7.2.1 High Evaporation

The next result shows that if $s > 2p$ then $L_i(t) \equiv R_i(t)$, $i = 1, 2$, is a GUAS solution of (3). In other words, when the evaporation rate is high and the pheromones cannot accumulate, then the positive feedback process leading to a favorable trail cannot take place and eventually the traffic will be divided equally along the two possible branches.

Theorem 1 If $s > 2p > 0$ then $\mathbf{0}$ is a GUAS solution of (4) for any $\tau > 0$ and $r \geq 1$.

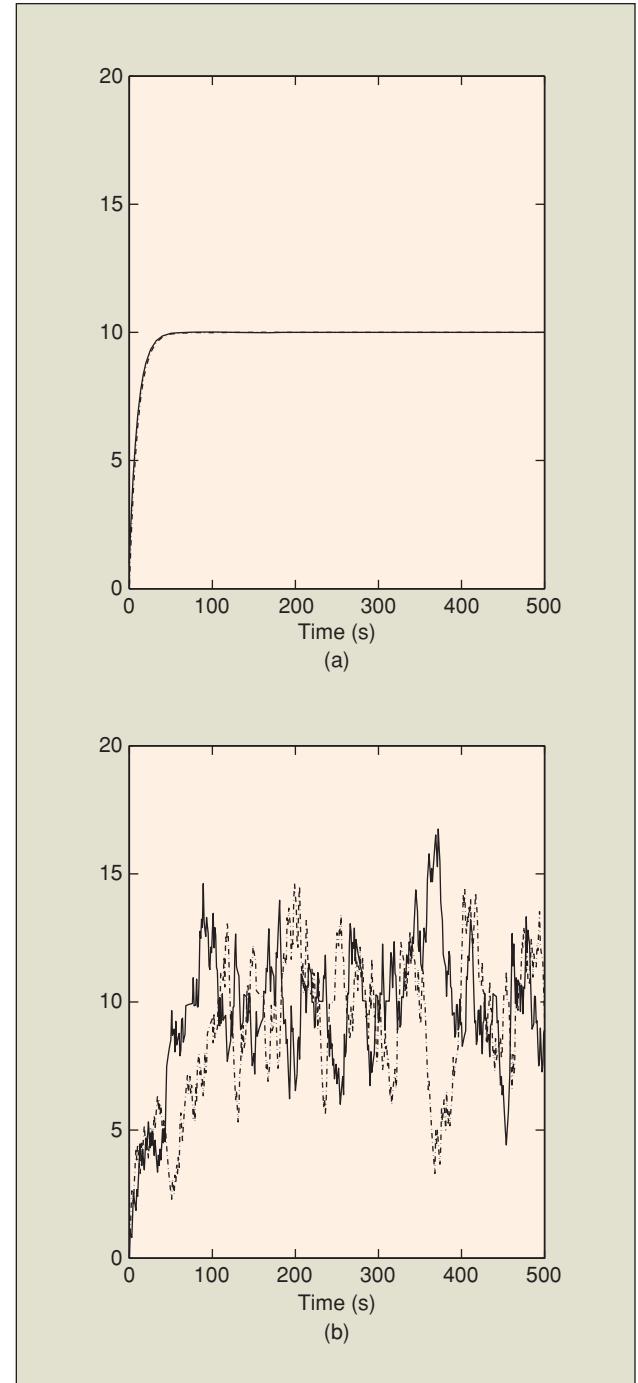


FIGURE 6 $L_1(t)$ (solid line) and $R_1(t)$ (dashed line) for $r = 2$, $s = 0.1$. (a) Solution of DDE (3) with $q = 0.016$, (b) Monte Carlo simulation.

7.2.2 Low Evaporation

For $s \in (0, 2p)$ the system admits three equilibrium solutions. **Proposition 2** If $s \in (0, 2p)$ then $\mathbf{0}$ is an unstable solution of (4), and both \mathbf{v}^1 and $-\mathbf{v}^1$ are uniformly asymptotically stable solutions.

Thus, $L_1(t) - R_1(t) \equiv L_2(t) - R_2(t) \equiv F\mathbf{v}^1$ and $L_1(t) - R_1(t) \equiv L_2(t) - R_2(t) \equiv -F\mathbf{v}^1$ are stable solutions of the averaged model. In other words, for low evaporation the system has a tendency towards a non-symmetric state, where one trail is more favorable than the other.

Appendix: Proofs

Proof of Theorem 1. Let $z_i(t) := p v_i(t)$, $i = 1, 2$. Then (4) becomes

$$\dot{\mathbf{z}} = -s\mathbf{z} + p \tanh(\mathbf{z}(t)) + M \tanh(\mathbf{z}(t - \tau)) + M \tanh(\mathbf{z}(t - r\tau)), \quad (7)$$

where $\mathbf{z}(t) := (z_1(t), z_2(t))^T$, $\tanh(\mathbf{z}(t)) = (\tanh(z_1(t)), \tanh(z_2(t)))^T$, and $M = \begin{pmatrix} 0 & P/2 \\ P/2 & 0 \end{pmatrix}$.

Consider the Lyapunov-Krasovskii functional (see, e.g., [37, Ch. 1])

$$W(t, \mathbf{z}_t) := 2\mathbf{z}^T(t)\mathbf{z}(t) + (s - p) \int_{t-\tau}^t \tanh^T(\mathbf{z}(u)) \tanh(\mathbf{z}(u)) du \\ + (s - p) \int_{t-r\tau}^t \tanh^T(\mathbf{z}(u)) \tanh(\mathbf{z}(u)) du,$$

where $\mathbf{z}_t := \{\mathbf{z}(a) : a \in [t - r\tau, t]\}$. Differentiating W along the trajectories of (7) yields

$$\dot{W}(t, \mathbf{z}_t) = 4\mathbf{z}^T(t)(p \tanh(\mathbf{z}(t)) + M \tanh(\mathbf{z}(t - \tau)) + M \tanh(\mathbf{z}(t - r\tau))) \\ + 2(s - p) \tanh^T(\mathbf{z}(t)) \tanh(\mathbf{z}(t)) - 4s\mathbf{z}^T(t)\mathbf{z}(t) \\ + (p - s) \tanh^T(\mathbf{z}(t - \tau)) \tanh(\mathbf{z}(t - \tau)) \\ + (p - s) \tanh^T(\mathbf{z}(t - r\tau)) \tanh(\mathbf{z}(t - r\tau)).$$

It is easy to verify that $\tanh^T(\mathbf{x}) \tanh(\mathbf{x}) \leq \mathbf{x}^T \mathbf{x}$ and $\mathbf{x}^T(\tanh(\mathbf{x}) - \mathbf{x}) \leq 0$ for all \mathbf{x} , and using this yields $\dot{W}(t, \mathbf{z}_t) \leq -\mathbf{d}^T(t)N\mathbf{d}(t)$, where $\mathbf{d}(t) := (\mathbf{z}^T(t), \tanh^T(\mathbf{z}(t - \tau)), \tanh^T(\mathbf{z}(t - r\tau)))^T$, and

$$N := \begin{pmatrix} 2(s-p)I & -2M & -2M \\ -2M^T & (s-p)I & 0 \\ -2M^T & 0 & (s-p)I \end{pmatrix}.$$

The eigenvalues of N are $\lambda_1 = \lambda_2 = s - p$, $\lambda_3 = \lambda_4 = (3s - 3p + h)/2$, and $\lambda_5 = \lambda_6 = (3s - 3p - h)/2$, where $h := ((s-p)^2 + 8p^2)^{1/2}$. For $s > 2p$ all the λ_i 's are real and strictly positive, so N is a positive-definite matrix. Using the Lyapunov-Krasovskii stability theorem [37] completes the proof. \square

Proof of Proposition 2. We require the following result.

Proposition 3 Given $b \in \mathbb{R}$ and $a, \tau, r > 0$, consider the linear DDE

$$\begin{aligned} \dot{\eta}_1(t) &= (2a - b)\eta_1(t) + a(\eta_2(t - \tau) + \eta_2(t - r\tau)), \\ \dot{\eta}_2(t) &= (2a - b)\eta_2(t) + a(\eta_1(t - \tau) + \eta_1(t - r\tau)). \end{aligned} \quad (8)$$

If $b < 4a$ ($b > 4a$) then $\mathbf{0}$ is an unstable (GUAS) solution of (8).

Proof. We follow the reasoning in [36]. Consider a solution in the form $\eta(t) = \exp(\lambda t)\mathbf{c}$, where $\eta(t) := (\eta_1(t), \eta_2(t))^T$, and $\mathbf{c} := (c_1, c_2)^T$. This yields the characteristic equation

$$\begin{pmatrix} \lambda + b - 2a & -az \\ -az & \lambda + b - 2a \end{pmatrix} \mathbf{c} = \mathbf{0},$$

Note that both Theorem 1 and Proposition 2 are *delay-independent* results as they hold for any delay $\tau > 0$.

8. Conclusions

In many fields of science, researchers provided verbal

descriptions of various phenomena. FM is a simple and direct approach for transforming these verbal descriptions into well-defined mathematical models.

The development of such models can also be used to address various engineering problems. This is because many

where $z := \exp(-\lambda\tau) + \exp(-\lambda r\tau)$. A (nontrivial) solution exists if and only if

$$\Delta_+(\lambda)\Delta_-(\lambda) = 0, \quad (9)$$

where $\Delta_{\pm}(\lambda) := \lambda + b - 2a \pm az$. Substituting $\lambda = \mu + jw$, with $\mu, w \in \mathbb{R}$, and $j = \sqrt{-1}$, in (9) yields $\Delta_{\pm}(\lambda) = R_{\pm}(\mu, w) + jI_{\pm}(\mu, w)$, where

$$\begin{aligned} R_{\pm}(\mu, w) &:= \mu + b - 2a \pm a(\exp(-\tau\mu)\cos(\tau w) + \exp(-r\tau\mu)\cos(r\tau w)), \\ I_{\pm}(\mu, w) &:= w \pm (\exp(-\tau\mu)\sin(\tau w) + \exp(-r\tau\mu)\sin(r\tau w))(-a). \end{aligned} \quad (10)$$

Letting $R(\mu) := \mu + b - 2a - a(\exp(-\tau\mu) + \exp(-r\tau\mu))$ yields

$$R_+(\mu, w) \geq R(\mu), \quad \text{and} \quad R_-(\mu, w) \geq R(\mu). \quad (11)$$

Note that $R(0) = b - 4a$ and that $\dot{R}(\mu) > 1$, so $R(\mu) \geq b - 4a + \mu$ for all $\mu \geq 0$.

If $b > 4a$ then, $R(\mu) > 0$ for all $\mu \geq 0$. If $\lambda = \mu + jw$ is a solution of (9), then either $R_+(\mu, w) = I_+(\mu, w) = 0$ or $R_-(\mu, w) = I_-(\mu, w) = 0$. In both cases, (11) implies that $R(\mu) \leq 0$, so $\mu < 0$. Thus, the real part of every root of the characteristic equation is strictly negative, so the linear DDE is GUAS [41].

If $b < 4a$ then $\Delta_-(0) = b - 4a < 0$, and $\lim_{\mu \rightarrow +\infty} \Delta_-(\mu + j0) = +\infty$, for any $a, \tau, r > 0$. Hence, there exists $\mu > 0$ such that $\Delta_-(\mu) = 0$, i.e., the characteristic equation admits a real and positive root. \square

We can now prove Proposition 2. We begin by considering the behavior near $\mathbf{0}$. Linearizing (4) about $v_1 = v_2 = 0$ yields the linear DDE:

$$\begin{aligned} \dot{\eta}_1(t) &= (p - s)\eta_1(t) + (\eta_2(t - \tau) + \eta_2(t - r\tau))p/2, \\ \dot{\eta}_2(t) &= (p - s)\eta_2(t) + (\eta_1(t - \tau) + \eta_1(t - r\tau))p/2, \end{aligned}$$

and Proposition 3 implies that $\mathbf{0}$ is not stable when $s < 2p$.

To analyze the stability of $\mathbf{v}^1 = (\underline{v}, \underline{v})^T$, let $\mathbf{x}(t) := \mathbf{v}(t) - \mathbf{v}^1$. Then

$$\begin{aligned} \dot{x}_1(t) &= -s\underline{v} - s\eta_1(t) + \tanh(p(\underline{v} + x_1(t))) \\ &\quad + (\tanh(p(\underline{v} + x_2(t - \tau))) + \tanh(p(\underline{v} + x_2(t - r\tau))))/2, \\ \dot{x}_2(t) &= -s\underline{v} - s\eta_2(t) + \tanh(p(\underline{v} + x_2(t))) \\ &\quad + (\tanh(p(\underline{v} + x_1(t - \tau))) + \tanh(p(\underline{v} + x_1(t - r\tau))))/2. \end{aligned} \quad (12)$$

Linearizing (12) about $\mathbf{x} = \mathbf{0}$, and using the expansion $\tanh(p(x + y)) = \tanh(py) + p(1 - \tanh^2(py))x + o(x)$ yields

$$\begin{aligned} \dot{\eta}_1(t) &= 2\tanh(p\underline{v}) - s\underline{v} - s\eta_1(t) + c(2\eta_2(t) + \eta_2(t - \tau) + \eta_2(t - r\tau)), \\ \dot{\eta}_2(t) &= 2\tanh(p\underline{v}) - s\underline{v} - s\eta_2(t) + c(2\eta_1(t) + \eta_1(t - \tau) + \eta_1(t - r\tau)), \end{aligned}$$

where $c := (1 - \tanh^2(p\underline{v}))p/2$. Since \underline{v} is a solution of (5), this simplifies to

$$\begin{aligned} \dot{\eta}_1(t) &= (2m - s)\eta_1(t) + m(\eta_2(t - \tau) + \eta_2(t - r\tau)), \\ \dot{\eta}_2(t) &= (2m - s)\eta_2(t) + m(\eta_1(t - \tau) + \eta_1(t - r\tau)), \end{aligned}$$

where $m := (4 - s^2\underline{v}^2)p/8$. Eq. (5) implies that $\underline{v} < 2/s$, so $m > 0$. Let $g(x) := px\tanh^2(px) - px + \tanh(px)$. Then $g(0) = 0$ and $\dot{g}(x) > 0$ for all $x > 0$, so $g(\underline{v}) > 0$. Using this and (5) yields $s > 4m$, and Proposition 3 implies that $\eta = \mathbf{0}$ is GUAS. Hence, \mathbf{v}^1 is a uniformly asymptotically stable solution of (4). The analysis of $\mathbf{v}(t) \equiv -\mathbf{v}^1$ is similar and is therefore omitted. \square

The set of ants choosing the left (right) path corresponds to the state of the first (second) neuron. The effect of the chemical communication between the ants corresponds to the time-delayed feedback connections between the neurons.

artificial systems must function in the real world and address problems similar to those encountered by biological agents such as plants or animals. The field of biomimicry is concerned with developing artificial systems inspired by the behavior of biological agents. An important component in this field is the ability to perform reverse engineering of an animal's functioning, and then implement this behavior in an artificial system. We believe that the FM approach may be suitable for addressing biomimicry in a systematic manner. Namely, start with a verbal description of an animal's behavior (e.g., foraging in ants) and, using fuzzy logic theory, obtain a mathematical model of this behavior which can be implemented by artificial systems (e.g., autonomous robots).

In this paper, we took a first step in this direction by applying FM to transform a verbal description of the foraging behavior of ants into a mathematical model. Simulations and rigorous analysis of the resulting model show that it is congruent with the behavior actually observed in nature. Furthermore, when the fuzzy model is substituted in a mathematical model for the colony of foragers, it leads to an interesting connection with models used in the theory of artificial neural networks. Unlike previous models, the fuzzy model is also simple enough to allow a rather detailed analytical analysis.

The collective behavior of social insects inspired many interesting engineering designs (see, e.g., [24], [40]). Further research is needed in order to study the application of the model studied here to various engineering problems.

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